The Cone

Defnᵒ:- A cone is a surface generated by line which passes through a fixed point & satisfies one more geometrical condn like passes through a given curve surface.

The fixed point is called vertex of ~~curve~~ cone & given curve is called guiding curve of cone.

Any line through the vertex & guiding curve is called generator.

* Cone with Vertex at the Origin :-

The eqn of cone whose vertex is at origin is homogeneous of second degree in x, y, z & conversely.

eqn is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Example:- Necessary data:-

1) One second deg. eqn
2) One first deg. eqn

we know that every 2nd deg homo. eqn represents a cone with vertex origin.

∴ consider the given second deg eqn & using the given 1st deg. eqn make it homo.

Q. Find the eqn of a cone whose vertex is at origin & guiding curve is

$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1, \quad y=b.$

$\Rightarrow$ $y=b$ can be expressed as $\frac{y}{b}=1$

Now consider $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1.$

Now making eq^n homogeneous.

$$\Rightarrow \frac{x^2}{a^2} + \frac{z^2}{c^2} = (1)^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{z^2}{c^2} = \left(\frac{y}{b}\right)^2$$

Hence the required eq^n is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0.$$

* The dc's or dr's of a generator of cone whose vertex is the origin satisfy the eq^n of the cone.

$\Rightarrow$ Let $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — (1)

be eq^n of cone with vertex origin.

Let l, m, n be dc's or dr's of generator

$\therefore$ eq^n of generator will be

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} = K \text{ say.}$$

$\therefore x = lk,\ y = mk,\ z = nk$

(x, y, z) are co-ordinates of any point on generator

$\therefore$ put in (1).

$\therefore a(lk)^2 + b(mk)^2 + c(nk)^2 + 2f(mk)(nk) + 2g(nk)(lk) + 2h(lk)(mk) = 0$

$\therefore al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm = 0$

then we can say, line with dc's l, m, n is generator of the cone.

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

* Quadratic Cone through the Axes :-

passing through three co-ordinate axes is

$$fyz + gzx + hxy = 0.$$

Q. Find the eqn of the cone which passes through three co-ordinate axes as well as the two lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$

⇒ The eqn of cone through three co-ordinate axes is

$$fyz + gzx + hxy = 0 \quad \text{—— (1)}$$

The dir's of given two lines are,
$(1, -2, 3)$ & $(3, -1, 1)$.
& that satisfies given eqn (1).

$\therefore f(-2\times 3) + g(1\times 3) + h(1\times(-2)) = 0$

$\therefore -6f + 3g - 2h = 0 \Rightarrow \cancel{-3f}$

& $f(1\times(-1)) + g(1\times 3) + h(3\times(-1)) = 0$

$-f + 3g - 3h = 0$

$\therefore$ by cramers rule, solving,

$$\frac{f}{\begin{vmatrix} 3 & -2 \\ 3 & -3 \end{vmatrix}} = \frac{-g}{\begin{vmatrix} -6 & -2 \\ -1 & -3 \end{vmatrix}} = \frac{h}{\begin{vmatrix} -6 & 3 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{f}{-3} = \frac{-g}{16} = \frac{h}{-15}$$

$\therefore$ put f, g, h in (1) we get

$$-3yz - 16zx - 15xy = 0$$

or $3yz + 16zx + 15xy = 0$

which is required equation.

Q. Find the eqn of the cone whose vertex is $(1, 1, 3)$ & which passes through $4x^2 + z^2 = 1$, $y = 4$.

⇒ l, m, n are dr's satisfies the generator as $(1, 1, 3)$ is the vertex.

$\therefore$ eqn of generator is,

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-3}{n} \quad \text{--- (1)}$$

put $y = 4$

$\therefore \frac{x-1}{l} = \frac{3}{m} = \frac{z-3}{n}$

$\therefore$ x & z are,

$x = 1 + \frac{3l}{m}$ & $z = 3 + \frac{3n}{m}$

$\therefore$ given eqn is

$4x^2 + z^2 = 1$

$4\left(1 + \frac{3l}{m}\right)^2 + \left(3 + \frac{3n}{m}\right)^2 = 1.$

but from (1) we can put.

$\frac{l}{m} = \frac{x-1}{y-1}$ & $\frac{n}{m} = \frac{z-3}{y-1}$

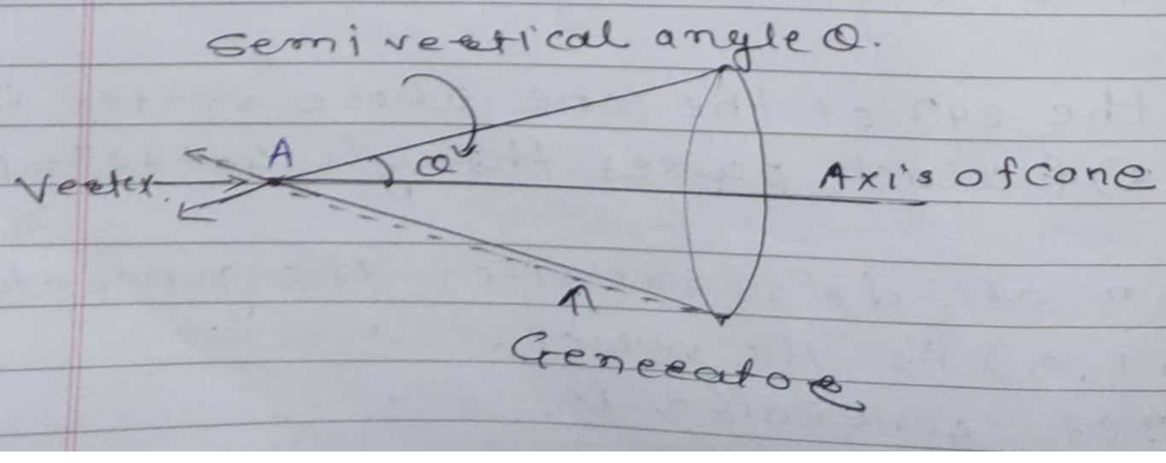
$\therefore$ eqn becomes

$4\left[1 + 3\left(\frac{(x-1)}{(y-1)}\right)\right]^2 + \left[3 + 3\left(\frac{z-3}{y-1}\right)\right]^2 = 1$

i.e $12x^2 + 4y^2 + 3z^2 + 6yz + 8xy - 32x - 34y - 24z + 69 = 0$

required eqn of cone.

* Right circular cone:-



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For solving problem on Right circular Cone:
① Co-ordinate of vertex A (a, b, c)
② dr's of axis l, m, n.
③ semivertical angle θ.

Q. Find the eqn of the Right circular cone whose vertex is $(1,-1,2)$ & axis is the line $\frac{(x-1)}{2} = \frac{y+1}{1} = \frac{z-2}{-2}$ & semi vertical angle 45°

⇒ Step 1: Let $P(x, y, z)$ be any point on generator

Step 2: Given vertex $(1,-1,2)$

$\therefore$ dr's of AP are $(x-1), y+1, z-2$

Step 3: dr's of axis $2, 1, -2$.

Step 4: Use formula for $\cos\theta$.

$$\therefore \cos 45^\circ = \frac{2(x-1)+1(y+1)-2(z-2)}{\sqrt{4+1+4}\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}}$$

Step 5: Squaring on both sides we get,

$$9[(x-1)^2+(y+1)^2+(z-2)^2] = 2(2x+y-2z+3)^2$$

$$5x^2+8y^2+13z^2-4xy+4yz+8xz-30x+12y-24z+45=0$$

* General Eqn of cone :-

Condition for general second deg. eqn to represent a cone & to find the co-ordinates of the vertex.

First check - eqn is homogenous of deg. second & we have seen that the vertex is origin

Now consider a eq^n

$$f(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy + 2ux + 2vy + 2wz + d = 0 \quad \text{—(1)}$$

Since the above eq^n is not homogeneous the vertex is at say (α, β, γ) (not at origin)

$\therefore$ We shift the origin to the vertex (α, β, γ) by the transformation

$x = X+\alpha$, $y = Y+\beta$, $z = Z+\gamma$ then eq^n (1) becomes.

$$f(X,Y,Z) = aX^2+bY^2+cZ^2+2fYZ+2gZX + 2hXY + 2(a\alpha+h\beta+g\gamma+u)X + 2(h\alpha+b\beta+f\gamma+v)Y + 2(g\alpha+f\beta+c\gamma+w)Z + (a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta + 2u\alpha+2v\beta+2w\gamma+d) = 0.$$

$\because$ we already take that it is homogeneous

$\therefore a\alpha+h\beta+g\gamma+u=0$

$h\alpha+b\beta+f\gamma+v=0$

$g\alpha+f\beta+c\gamma+w=0$ &

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta + 2u\alpha+2v\beta+2w\gamma+d=0$$

OR

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \quad \text{—(2)}$$

While solving the problems,

Take given eq^n $f(x,y,z)=0$

make it homogeneous by multiplying the terms by suitable power of variable a variable t, so that.

$$f(x,y,z,t) = ax^2+by^2+cz^2+2fyz+2gzx + 2hxy+2uxt+2vyt+2wzt+dt^2=0$$

& We find

$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0, \frac{\partial F}{\partial t}=0$ &

substitute back $t=1$

& Find value of x, y, z & if satisfies the eqn then required eqn is eqn of cone.

Q. Show that the eqn

$$2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$$

represent cone:-

⇒ Making the ~~above~~ eqn homogeneous.

$$f(x,y,z,t) = 2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$$

i.e

$$= 2x^2+2y^2+7z^2-10yz-10zx+2xt+2yt+26zt-17t^2$$

Now from following equations:

$$\frac{\partial F}{\partial x} = 4x-10z+2t=0 \quad -① $$

$$\frac{\partial F}{\partial y} = 4y-10z+2t=0 \quad -② $$

$$\frac{\partial F}{\partial z} = 14z-10y-10x+26t=0 \quad -③ $$

$$\frac{\partial F}{\partial t} = 2x+2y+26z-34t=0 \quad -④ $$

Taking $T=1$ & solving ①, ② & ③.

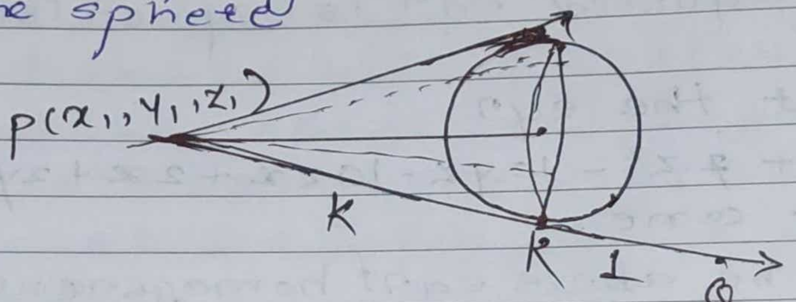
$\therefore$ $4x-10z+2=0$

$4y-10z+2=0$

$14z-10y-10x+26=0$.

we get $x=2,\ y=2\ \&\ z=1$
which satisfies (4)
$\therefore$ given eqn represent a cone whose vertex is $(2,2,1)$

* Enveloping cone:-
The locus of tangent lines from a given point to a sphere is called enveloping cone from the point to the sphere



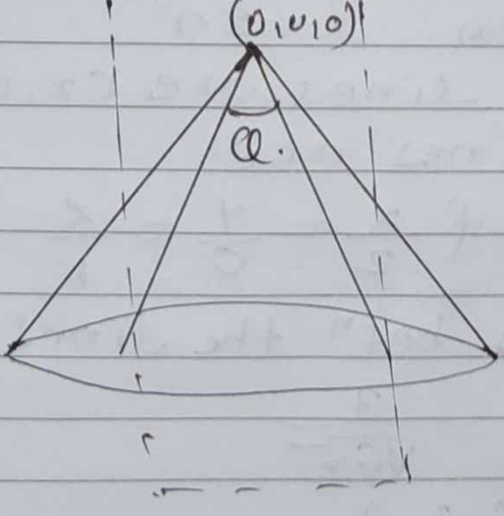
a] The eqn of enveloping cone from the point x_1, y_1, z_1 to the sphere
$x^2+y^2+z^2=a^2$ is given by
$SS_1=T^2$
where, $S \equiv x^2+y^2+z^2-a^2$
$S_1 \equiv x_1^2+y_1^2+z_1^2-a^2$
$T \equiv xx_1+yy_1+zz_1-a^2$

b] The eqn of the enveloping cone from (x_1, y_1, z_1) to $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is
also $SS_1=T^2$ where
$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d$
$S_1 \equiv x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$
$T \equiv xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$

Q. Find the eqn of a cone with vertex at (1,1,1) & generators touching the sphere
$x^2+y^2+z^2-2x+4z=1$

→ The point (x_1, y_1, z_1) is $(1,1,1)$
eqn of cone is
$SS_1 = T^2$
$S \equiv x^2+y^2+z^2-2x+4z-1$
$S_1 = x_1^2+y_1^2+z_1^2-2x_1+4z_1-1$
$= 1^2+1^2+1^2-2(1)+4(1)-1$
$= 4$, Value of $u=1$, $w=2$, $v=0$
$T \equiv xx_1+yy_1+zz_1-(x+x_1)+2(z+z_1)-1$
$= x+y+z-(x+1)+2(z+1)-1$
$= \cancel{x}+y+z-\cancel{x}-\cancel{1}+2z+\cancel{2}-\cancel{1}$
$= y+3z$
$SS_1 = T^2$
$4(x^2+y^2+z^2-2x+4z-1) = (y+3z)^2$
$4x^2+4y^2+4z^2-8x+16z-4 = y^2+6yz+9z^2$
$\Rightarrow 4x^2+3y^2-5z^2-6yz-8x+16z-4=0.$

Q. Angle betn the lines in which a plane through the vertex cuts a cone:-



Q. Find the angle between the lines in which the plane $x+3y-2z=0$ meets the cone $x^2+9y-4z^2=0$.

Let the eq^n of the line in which the plane cuts the cone be

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

Then since the lines lies on the plane $x+3y-2z=0$
it will be perpendicular to the normal d.r's are $1, 3, -2$

$\therefore \quad l+3m-2n=0$ —— (1)

Also since the d.r's of any generator of cone with vertex at the origin satisfies eq^n of cone itself we have,

$l^2+9m^2-4n^2=0$ —— (2)

From (1) we have

$2n = l+3m$

Substitute in eq^n (2)

$l^2+9m^2-(l+3m)^2=0$

$\Rightarrow -6ml=0$

i.e $l=0$ or $m=0$

If $l=0$ then $3m = 2n$

$\therefore$ The d.r's of line can be $(0,2,3)$

If $m=0$ then $l=2n$

then d.r's of lines are $(2,0,1)$

$\therefore$ The two lines are

$$\frac{x}{0} = \frac{y}{2} = \frac{z}{3} \quad \& \quad \frac{x}{2} = \frac{y}{0} = \frac{z}{1}$$

If θ is angle bet^n the lines then

$$\cos\theta = \frac{3}{\sqrt{13}\sqrt{5}} = \frac{3}{\sqrt{65}}$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{65}}\right).$$